Stroboscopic high-order nonlinearity in quantum optomechanics

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High-order quantum nonlinearity is an important prerequisite for the advanced quantum technology leading to universal quantum processing with large information capacity of continuous variables. We devise a method of stroboscopic application of a highly nonlinear potential to an initial squeezed thermal state of a mechanical oscillator. The mechanical states generated by the protocol clearly exhibit nonclassicality and the squeezing of a nonlinear quadrature, proving the higher-order quantum nonlinearity and rendering them a useful resource for the mechanical quantum technology. We analyze the main sources of decoherence and estimate possible achievable nonlinearities in the systems that are within reach. We test the method numerically on the cubic potential using the relevant parameters of a typical levitated optomechanical experiment and prove its feasibility.

Introduction.—Quantum Information Processing with continuous variables (CVs) [1] has achieved noticeable progress recently, particularly in quantum state engineering and communication [2]. A potential advantage of CVs is that the in principle unlimited information capacity of the infinitely-dimensional Hilbert space can be accessed by a homodyne detection [3]. In order to fully gain the benefits of CVs and to potentially access the universal quantum computation one at least requires a nonlinear cubic potential [4, 5]. Moreover, the CV quantum information processing can be greatly simplified and stabilized if the higher-order potentials are available [6]. A straightforward way to achieve the nonlinearity is to induce controllable nonlinear force on a linear oscillator. An alternative method to access the nonlinear gate for the quantum circuits is to use a measurement-based strategy assisted by an ancilla state prepared in such nonlinear quantum process [7–14]. A promising candidate to provide the key element for both methods is the field of optomechanics [15] that focuses on the systems in which radiation pressure of light or microwaves drives the mechanical motion. The optomechanical systems have reached a truly quantum domain demonstrating the effects ranging from the ground state cooling [16] and squeezing [17] of the mechanical motion to the entanglement of distant mechanical oscillators [18]. Of particular interest are the levitated systems in which the trapping potential of the mechanical motion is provided by an optical tweezer [19–21]. Such systems have proved useful in force sensing [22, 23], studies of quantum thermodynamics [24–26], testing fundamental physics [27–29] and probing quantum gravity [30, 31]. From the technical point of view, the levitated systems have recently demonstrated strong progress in the controllability and engineering, particularly, cooling towards the ground state [32–35]. Besides the inherently nonlinear optomechanical interaction met in the standard bulk optomechanical systems the levitated ones possess the attractive possibility of engineering the nonlinear trapping potential [25, 36–40]. Moreover, the trapping potentials can be made time-dependent and manipulated at rates higher than the mechanical decoherence one and even the mechanical frequency [41].

In the present letter we propose a high-order nonlinearity for optomechanical systems with time variable external force. We theoretically investigate the dynamics of a levitated nanoparticle in presence of simultaneously a harmonic and a strong stroboscopically applied nonlinear potentials enabled by the engineering of the trapping beam. Using Suzuki-Trotter expansion [42] we induce the simultaneous action of the potentials and obtain the Wigner functions of the quantum motional states achievable in this system. We directly observe very nonclassical negative Wigner function [7, 8] generated by highly nonlinear quantum mechanics. The oscillations of Wigner function reaching negative values witness the quantum dynamics required for nonlinear phase gate. We prove a nonlinear combination of the canonical quadratures of the mechanical oscillator to be squeezed below the ground state variance that is an important prerequisite of this state being a resource for the measurement-based quantum computation [11, 13]. For this method, we focus our attention to realistic versions of nonlinear phase states, namely the cubic phase state. The method allows straightforward extension to more complex nonlinear potentials which can be used to flexibly generate other resources for nonlinear gates and their applications [6, 13].

Nonlinear phase dynamics.—An ideal action of highly nonlinear potential $V(x)$ modifies the phase of quantum state by unitary evolution operator $\exp[iV(x)\tau]$ (nonlinear phase gate), where τ is the duration of the evolution in the potential. A nonlinear phase state (particularly, the cubic phase state introduced in [5]) as the outcome of evolution in a nonlinear potential $V(x;\Gamma)$ is defined as

$$|\gamma_V\rangle \propto \int \mathrm{d}x\, e^{iV(x;\Gamma)\tau} |x\rangle , \tag{1}$$

where $V(x;\Gamma) = \Gamma x^k,\ k \geq 3$, is a highly nonlinear potential, $|x\rangle$ the position eigenstate $\hat{x}|x\rangle = x$. We distinguish between the rate Γ entering the expression for the potential and the gain $\gamma \equiv \Gamma\tau$ showing the strength of the resulting nonlinearity. The state (1) requires an infinite squeezing of the ideal ground state before the nonlinear potential is applied. More physical is an approximation of this state obtained from a finitely squeezed thermal state, ideally, vacuum, by the application of V:

$$\rho(V, r, n_0) = e^{iV(x)\tau} S(r)\rho_0 S^\dagger(r) e^{-iV(x)\tau} \tag{2}$$

where $S(r) = \exp[i(a^2 r - a^{\dagger 2} r^*)]$ is a squeezing operator, and the initial state ρ_0 is thermal with mean occupation n_0. An important aspect of Eq. (2) is that the evolution operator $\exp[iV(x)\tau]$ of the nonlinear phase gate describes nondemolition quantum dynamics with an invariant position variable x. Such physical system has to have a negligible free evolution to reach the limit of a nonlinear quantum state. A non-negligible free evolution, unavoidable in presence of harmonic trap, causes degradation of the nonlinearity. To alleviate the influence of the free evolution we propose to apply the nonlinear potential stroboscopically. That is we consider the regime in which in addition to the harmonic trap a nonlinear potential is switched on for a short interval once per a harmonic oscillation. We investigate the achievable nonlinearities and the influence of the mechanical damping and decoherence on the nonlinear state preparation.

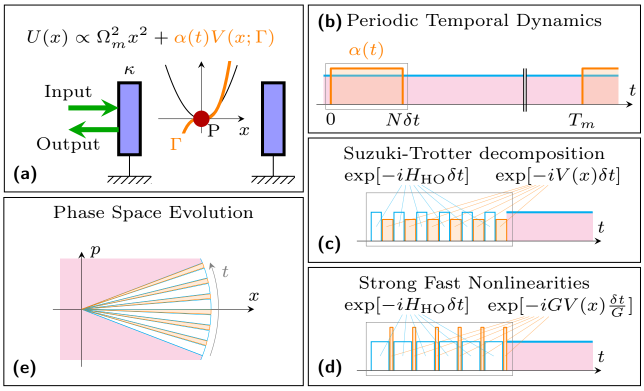

FIG. 1. (a) A levitated optomechanical system. A dielectric subwavelength particle (P) is trapped by a tweezer (not shown) within a high-Q cavity κ. The particle feels a total potential $U(x)$ that is a sum of the harmonic and the nonlinear parts, both provided by the trapping beam. The particle can be probed by the laser light. (b) The nonlinear part of the potential is switched on for only a fraction of the mechanical period. Such an evolution can be approximated by a sequence of pulsed interactions as in (c,d). Orange sectors represent action of the nonlinear potential, cyan sectors without fill represent the harmonic evolution, pink filled sector denotes damped harmonic evolution. (e) Phase space picture of the evolution over a single mechanical period given by Eq. (6).

The nonlinear stroboscopic protocol.—To implement the stroboscopic method, it is possible to use a levitated nanoparticle [41], a mirror equipped with a fully-optical spring [43], or a membrane with electrodes allowing its nonlinear actuation and driving [44]. Any of such systems can be posed into an artificial nonlinear potential $V(x)$, particularly, the cubic potential $V_3(x) \propto x^3$ for the pioneering test. In this manuscript we focus on the levitated nanoparticles, although the principal results remain valid for the other systems as well.

The mechanical mode is a harmonic oscillator of eigen frequency Ω_m, described by position and momentum quadratures, respectively, x and p, such that $[x, p] = 2i$. The oscillator is coupled to a thermal bath at rate η_m. We also assume stroboscopic application of an external nonlinear potential $\alpha(t)V(x)$ with a piecewise constant $\alpha(t)$ illustrating the possibility to periodically switch the potential on and off. The Hamiltonian of the system, therefore, reads ($\hbar = 1$)

$$H = H_{\rm HO} + \alpha(t)V(x), \quad H_{\rm HO} = \frac{1}{4}\Omega_m(x^2 + p^2), \tag{3}$$

In the case of absent mechanical damping and decoherence the unitary evolution of the oscillator is given by $\rho(t) = \mathcal{U}(t, t_0)\rho(t_0)\mathcal{U}^\dagger(t, t_0)$, with $\mathcal{U}(t + \delta t, t) = \exp[-iH\delta t]$. We split the Hamiltonian into the free evolution and the nonlinear terms, and use the Suzuki-Trotter expansion [42] for $\mathcal{U}$ to obtain for a top-hat function $\alpha(t) = \Pi(0, N\delta t)$ (cf. Fig. 1 (b))

$$\begin{aligned}\mathcal{U}(t + N\delta t, t) &= \Big[\exp[-i(H_{\rm HO} + V(x))\delta t]\Big]^N \\ &\approx \Big[\mathcal{U}_{\rm HO}(\delta t)\mathcal{U}_{\rm NL}(\delta t) + O(\delta t^2)\Big]^N, \end{aligned}\tag{4}$$

where $\mathcal{U}_{\rm HO}(\delta t) \equiv \exp(-iH_{\rm HO}\delta t)$, $\mathcal{U}_{\rm NL}(\delta t) \equiv \exp(-iV(x)\delta t)$, N is called the Trotter number. The nonlinearity is switched on for only a fraction of the mechanical period, so that $N\delta t \ll T_m \equiv 2\pi/\Omega_m$. Thereby the simultaneous action of the free rotation and the nonlinearity in experiment can be approximated by their sequential action evaluated over short periods of time δt (see Fig. 1 (c)). Note that the approximate expansion Eq. (4) is simultaneously the exact solution corresponding to a sequence of strong instantaneous nonlinear pulses with the fixed product $V(x)\delta t$ interleaving the harmonic evolution. That is, in the limit $\delta t \to 0$ with fixed $\gamma = \Gamma\delta t$, converging to $\alpha(t) = \sum_{i=1}^N \delta(t - i\delta t)$, see Fig. 1 (d).

Ideally, in absence of the free evolution $\mathcal{U}_{\rm HO}$, the repetitive application of a weaker nonlinear potential could asymptotically lead to the effect indistinguishable from that of a strong nonlinear potential applied for shorter time. That is, in that case, $\mathcal{U}(t + N\delta t, t) = \mathcal{U}_{\rm NL}(N\delta t)$. In practice, however, the repetitive application will face deterioration caused by the free evolution of the oscillator and the coupling to thermal bath. The free evolution mixes together position and momentum variables and changes the nondemolition dynamics from that purely determined by $V(x)$. Focusing to reach the desired form of the evolution $\exp[iV(x)\tau]$, the nonlinearity that can be achieved in one period of harmonic oscillations is limited.

A method to avoid the free evolution would be to precisely time the individual pulses of nonlinear potential in such a way that the sequential applications happen exactly in the same phase of the sequential harmonic oscillations once per period. This however can be done for a limited number of oscillations determined by the impact of thermal environment. We account for the latter by assuming for the rest of the mechanical period damped evolution described by the Langevin equations

$$\dot{x} = \Omega_m p; \quad \dot{p} = -\Omega_m x - \eta_m p + \sqrt{2\eta_m}\xi, \tag{5}$$

where ξ is the quantum Langevin force, obeying $[\xi(t), x(t)] = i\sqrt{2\eta_m}$ and $\frac{1}{2}\langle\{\xi(t), \xi(t')\}\rangle = (2n_{\rm th}+1)\delta(t-t')$ with $n_{\rm th}$ being the occupation of the bath. The density matrix of the particle after one period of oscillations including the action of the nonlinear potential and subsequent damping can be evaluated as

$$\rho(T_m) = \mathcal{D}_N[\rho(0)] = \\ \mathrm{Tr}_B\left[\mathcal{U}_B\mathcal{U}(N\delta t, 0)\Big(\rho(0)\otimes\rho_B\Big)\mathcal{U}^\dagger(N\delta t, 0)\mathcal{U}_B^\dagger\right], \tag{6}$$

where ρ_B is the thermal state of the bath, $\mathcal{U}_B$ is the joint particle-bath evolution operator, and Tr_B means the trace operation performed over the bath variables.

The Eq. (6) approximates the damped evolution of an oscillator in a nonlinear potential by a sequence of individual harmonic, nonlinear and damped harmonic evolutions. Each of this transformations can be evaluated numerically as follows. First, we start from a squeezed thermal state $\rho(0)$, which has a representation by the Wigner function in the phase space

$$W_{\rm th}(x, p; n_0, s) = \frac{\exp\left(-\frac{1}{2}\left[\frac{(x/s)^2+(ps)^2}{2n_0+1}\right]\right)}{2\pi(2n_0+1)}. \tag{7}$$

The Wigner function corresponding to a quantum state ρ is defined [45] as

$$W(x, p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipy}\langle x+y|\,\rho\,|x-y\rangle\,, \tag{8}$$

and the corresponding density matrix element can be obtained from the Wigner function by an inverse Fourier transform. It is therefore possible to extend this approach to any $W(x, p)$ beyond the Gaussian states. The evolution in the nonlinear potential can be straightforwardly computed in the position eigenbasis:

$$\langle x|\,\mathcal{U}_{\rm NL}(\delta t)\rho(0)\mathcal{U}_{\rm NL}^\dagger(\delta t)\,|x'\rangle = \langle x|\,\rho(0)\,|x'\rangle\, e^{-i[V(x)-V(x')]\delta t}. \tag{9}$$

The undamped harmonic evolution is most easily represented by the rotation of Wigner function in the phase space. A unitary rotation for an angle $\theta = \Omega_m\delta t$ in the phase space maps the initial WF W_i onto the final W_f as

$$W_f(x, p) = W_i(x\cos\theta - p\sin\theta, p\cos\theta + x\sin\theta). \tag{10}$$

Damped harmonic evolution of a high-Q harmonic oscillator can also be evaluated in the phase space as a convolution of the initial Wigner function W_i with a thermal kernel

$$W_f(x, p) = \iint_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{d}v\, W_i(x-u, p-v)W_B(u, v), \tag{11}$$

where the expression for the kernel reads

$$W_B(u, v) = \frac{1}{2\pi\sigma_{\rm th}}\exp\left[-\frac{u^2+v^2}{2\sigma_{\rm th}}\right], \tag{12}$$

with $\sigma_{\rm th} = (2n_{\rm th}+1)2\pi\eta_m/\Omega_m$, where $n_{\rm th} \approx k_B T/(\hbar\Omega_m)$ is the thermal occupation of the bath set by its temperature T. The detailed derivation of Eq. (11) can be found in [46].

Using these techniques, one can evaluate the action of the $\mathcal{D}_N$ defined by Eq. (6) on the state of the quantum oscillator. This yields the quantum state of the particle after one mechanical oscillation. Repeatedly applying the same operations, one can obtain the state after M_T periods of the mechanical oscillations. Our purpose is then to explore the limits on the number of pulses N that can be applied within a single mechanical period, and the number of mechanical periods M_T, given certain figures of merit.

Application to the cubic nonlinearity.—For the rest of the present letter we illustrate the devised method numerically evaluating the evolution of a levitated particle in a cubic potential $V(x) = \Gamma x^3$. The quantum state obtained as a result of the considered sequence of interactions approximates the ideal state given by Eq. (1). The quality of the approximation can be assessed by evaluating the variance of a nonlinear quadrature $p - \lambda x^2$, or the cuts of the Wigner functions of the states. A reduction in nonlinear quadrature variance below the vacuum is a necessary condition for application of these states in nonlinear gates [11, 13]. On the other hand, the cut of Wigner function is very sensitive measure of quality of the states used in the recent experiments [47–50]. Fidelity is not a good measure of the success of the preparation of the quantum state [51] because it does not predict neither applicability of these states as resources not their highly nonclassical aspects.

A noise reduction in the cubic phase gate can be a relevant first test of the quality of our method. A state Eq. (2) should possess arbitrary high squeezing in the variable $p - \lambda x^2$ for $n_0 = 0$ given sufficient squeezing of the initial mechanical state. The approximate cubic state obtained from vacuum (that is, the state Eq. (2) with $n_0 = 0, s = 1$) has the following variance of the nonlinear quadrature

$$\sigma_3 \equiv \left\langle(p-\lambda x^2)^2\right\rangle - \left\langle p-\lambda x^2\right\rangle^2 = 1 + 2(\lambda - 3\gamma)^2, \tag{13}$$

The value $\sigma_3(\lambda)$ possesses an apparent minimum equal to the shot noise variance at $\lambda = 3\gamma$ for a given nonlinearity γ.

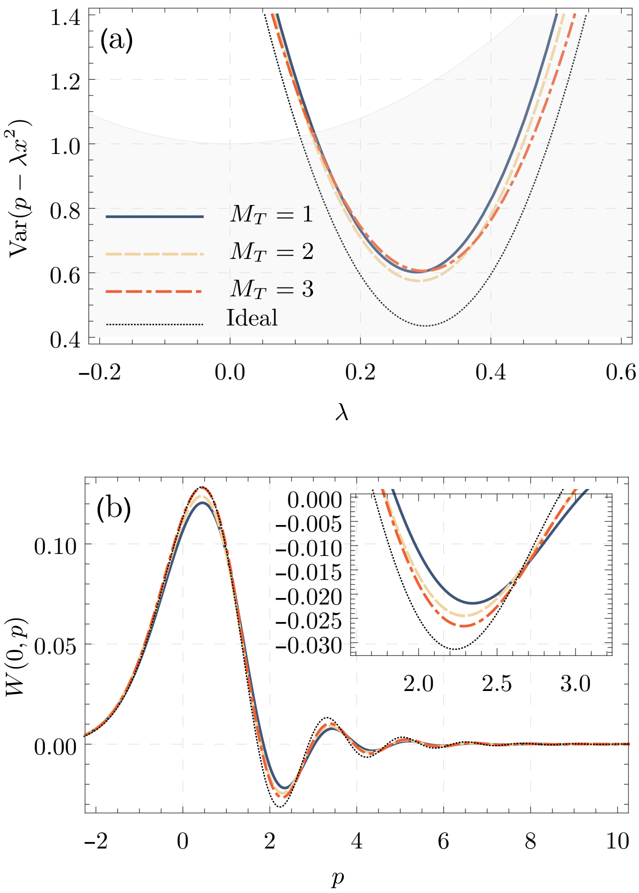

FIG. 2. Analysis of the stroboscopic protocol performance for the cubic potential $V(x) \propto x^3$. The nonlinear squeezing (a) and the Wigner function cuts (b) of the approximate nonlinear states obtained by repeated application of $\mathcal{D}_N$ (see Eq. (6)) to a squeezed thermal state (7) with $s = 1.6$ and $n_0 = 0.05$. The nonlinearity gain $\gamma = 0.05$. Different colors show different numbers M_T of mechanical periods involved. Thin dotted line shows the variance computed for the state Eq. (2) with the same nonlinear gain γ without free rotation and decoherence. (a) Nonlinear variance $\mathrm{Var}(p-\lambda x^2)$ as a function of λ for the states obtained by running the protocol for different numbers of mechanical periods. Thin gray line shows variance computed at vacuum state, values below (filled area) provide advantage over vacuum as a resource for implementation of a measurement-based gate [13]. (b) Wigner function cuts $W(0,p)$ of the same states. The cuts clearly show nonclassicality via negative values.

An important threshold is the variance of the nonlinear quadrature attained at the vacuum state

$$\sigma_3^{\mathrm{vac}} \equiv \left\langle (p-\lambda x^2)^2 \right\rangle_{|0\rangle} - \left\langle p-\lambda x^2 \right\rangle_{|0\rangle}^2 = 1 + 2\lambda^2; \quad (14)$$

For an initially squeezed thermal state the variance can be suppressed below the value given by Eq. (14). For the analysis of the role of the state presqueezing, see [46].

Our estimations of the nonlinear squeezing are illustrated by Fig. 2 (a). To produce the figure we estimate the nonlinear states that can be obtained with the proposed protocol for different numbers of mechanical periods M_T and different Trotter numbers N in accordance with Eq. (4). We use the combinations for which the total nonlinear gain, equal to the product $M_T N \Gamma \delta t = \gamma$, equals a certain value. For Fig. 2 we use $\gamma = 0.05$, combinations (M_T, N) for which $M_T \times N = 24$, and $\delta t = 1^\circ/\Omega_m$.

Importantly, application of the protocol allows to obtain the squeezing in the nonlinear quadrature below the shot-noise level even if the initial state of the particle is not pure. The nonlinear state created in a stroboscopic protocol clearly outperforms as a resource the vacuum for which the bound (14) holds. Moreover, the stroboscopic states approximate the one defined by Eq. (2) obtained in absence of the free rotation and thermal decoherence. The increase of the number of the mechanical periods M_T means the decrease of the duration of the free evolution $N\delta t$ within each of the periods which allows to obtain the desired lower values of the nonlinear variance. At the same time, increase of M_T leads to overall longer evolution and thereby stronger influence of the mechanical environment, therefore, there exists an optimal value of M_T allowing for the best nonlinear squeezing. For stronger nonlinearities Γ the worsening of the nonlinearity due to the free rotation within a single mechanical period is more pronounced, so that the optimal number M_T is higher. However, with higher occupations of the bath the effect of the latter can be tolerated for fewer mechanical oscillations, so the optimal M_T is lower.

The Wigner function of cubic phase state, i.e. the state given by Eq. (1) for $V(x;\gamma/\tau) = x^3\gamma/\tau$, reads [5, 8]

$$W_{\mathrm{CPS}}(x,p) \propto \mathrm{Ai}\left[\left(\frac{4}{3\gamma}\right)^{1/3}(3\gamma x^2 - p)\right], \quad (15)$$

where $\mathrm{Ai}(x)$ is the Airy function. This state with apparent non-Gaussian shape in the phase space exhibits fast oscillations in the positive momentum for any $\gamma > 0$. The Wigner functions of the states obtained by application of the stroboscopic protocol approach the one of Eq. (2). This is illustrated in Fig. 2 (b). The stroboscopic method can prepare the Wigner function approaching one from ideal cubic phase gate.

Indeed we see, that the produced states exhibit strong nonclassicality via the negative values of the Wigner function. In terms of the cuts $W(0,p)$ the strongest derogatory effect is the thermal noise coming from the bath, so the fewer number of the mechanical periods is used, the closer the corresponding cut is to the optimal one shown in dashed line. This is in contrast to the nonlinear squeezing, for which there is a visible trade-off between the number of mechanical periods utilized and the fraction of each of the periods during which the nonlinearity is kept switched on. In case of the lower thermal occupancy of the environment, the cuts also show a trade-off between the free rotation and the thermal decoherence.

This seeming inconsistency proves that the cuts $W(0,p)$ are an insufficient measure of the quality of the obtained nonlinear state as a resource for the measurement-based computation.

Conclusion and outlook.—We have devised a way to effectively prepare a nonlinear motional quantum state of a mechanical oscillator in an optomechanical cavity. The method has been numerically tested and proven useful for a particular case of the cubic nonlinearity. We have shown the method to work for the parameters inspired by recent results demonstrated by the levitated optomechanical systems [52, 53]. The optical trap with a cubic potential has been already used in the experiments [39, 40]. Levitated systems have recently shown noticeable progress towards ground state cooling [34, 35] and feedback-enhanced operation [33] which lays solid groundwork of the success of the proposed protocol. Its experimental implementation can demonstrate preparation of a strongly nongaussian quantum motional state. Further analysis of such a state will require either a full state tomography or better suited well-tailored methods to prove the nonclassicality [54, 55]. The optical read out can be improved using squeezed states of light [56]. This experimental step will open applications of the proposed method to other nonlinear potentials relevant for quantum computation [6, 9, 11, 13], quantum thermodynamics [57, 58] and quantum force sensing [59, 60].

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Supplemental Material: Stroboscopic high-order nonlinearity in quantum optomechanics

Impact of the thermal noise.—Owing to recent progress in design and manufacturing of the nanomechanical devices, the thermal noise is strongly suppressed. In this section we check the robustness of our scheme to the thermal noise from the environment and show that for the state-of-the art systems it is not hampering the quantum performance.

The thermal noise can be included by writing the standard Langevin equations:

$$\dot{x} = \Omega_m p; \quad \dot{p} = -\Omega_m x - \eta_m p + \sqrt{2\eta_m}\xi, \tag{S1}$$

where ξ is the quantum Langevin force, obeying $[\xi(t), x(t)] = i\sqrt{2\eta_m}$ and $\frac{1}{2}\langle\{\xi(t), \xi(t')\}\rangle = 2n_{\text{th}} + 1$.

The solution of these equations corresponding to a one full period of the mechanical oscillations ($T_m = 2\pi/\Omega_m$) for the experimentally relevant regime of high-Q mechanical oscillator ($Q \equiv \Omega_m/\eta_m \gg 1$) reads

$$x' = e^{-\eta_m T_m/2}\left[(\cos\zeta + \epsilon\sin\zeta)x(0) + \frac{1}{\sigma}\sin\zeta p(0) + \delta x(T_m)\right], \tag{S2}$$

$$p' = e^{-\eta_m T_m/2}\left[(\cos\zeta - \epsilon\sin\zeta)p(0) + \frac{1}{\sigma}\sin\zeta x(0) + \delta p(T_m)\right], \tag{S3}$$

with $\zeta = \Omega_m T_m \sigma$, $\sigma = \sqrt{1 - (\eta_m/2\Omega_m)}$ and $\epsilon = \eta_m/(2\Omega_m\sigma)$. The definitions for the noise operators $\delta x, \delta p$ read

$$\delta x(T_m) = \sqrt{2\eta_m}\int_0^{T_m} \mathrm{d}t\, e^{\eta_m t/2}\sin\Omega_m\sigma(T_m - t)\xi(t), \tag{S4}$$

$$\delta p(T_m) = \sqrt{2\eta_m}\int_0^{T_m} \mathrm{d}t\, e^{\eta_m t/2}\left[\cos\Omega_m\sigma(T_m - t) - \epsilon\sin\Omega_m\sigma(T - t)\right]\xi(t). \tag{S5}$$

These are the canonical quadratures of a mode in a thermal state with variance $\sigma_{\text{th}} = 2n_{\text{th}} + 1$.

The transformation of the Wigner function can be found as follows. Consider that at instant 0 the mechanical oscillator has WF $W_m(x(0), p(0))$. The mode of the bath is in a thermal state with WF

$$W_B(\delta x, \delta p; \sigma_{\text{th}}) = \frac{1}{2\pi\sigma_{\text{th}}}\exp\left[-\frac{\delta x^2 + \delta p^2}{2\sigma_{\text{th}}}\right]. \tag{S6}$$

Assuming the joint evolution of the mechanical oscillator and bath to be unitary, one can write the WF of the composite system (mechanical oscillator+bath)

$$W(x', p', \delta x, \delta p) = W_m(x[x', p', \delta x, \delta p], p[x', p', \delta x, \delta p]) \times W_B(\delta x, \delta p). \tag{S7}$$

Here the first argument of W_m, $x[\dots]$, means the solution of Eqs. (S2) and (S3) for x etc. Since in the contemporary experiments the quality of the mechanical oscillators can exceed $Q = 10^6$ (see e.g. [S61]), one can approximate $\sigma \approx 1$, $\epsilon = 0$, $\zeta = 2\pi$. Moreover, $\exp[-\eta_m T_m/2] \approx 1$. Therefore

$$x \approx x' - \sqrt{\frac{2\pi\eta_m}{\Omega_m}}\delta x \equiv x' - \theta\delta x, \quad p \approx p' - \theta\delta p, \quad \theta \equiv \sqrt{\frac{2\pi\eta_m}{\Omega_m}}. \tag{S8}$$

Thus to obtain the WF of the mechanical mode after this evolution, one has to trace out the degrees of freedom of the environment

$$W'(x', p') = \iint \mathrm{d}(\,\delta x)\mathrm{d}(\,\delta p)W_m(x' - \theta\delta x, p' - \theta\delta p)W_B(\delta x, \delta p; \sigma_{\text{th}}). \tag{S9}$$

Making a substitution $(u, v) = \theta\cdot(\delta x, \delta p)$ we arrive to the simple expression

$$W'(x', p') = \iint \mathrm{d}u\,\mathrm{d}v\, W_m(x' - u, p' - v)W'_B(u, v), \tag{S10}$$

where $W'_B(u, v) = W_B(u, v, \sigma_{\text{th}}\theta^2)$.

The Eq. (S10) describes a convolution of the initial Wigner function W_m with a WF of a thermal state, whose variance is reduced by the mechanical Q-factor. After rescaling this WF transforms into a very narrow Gaussian with

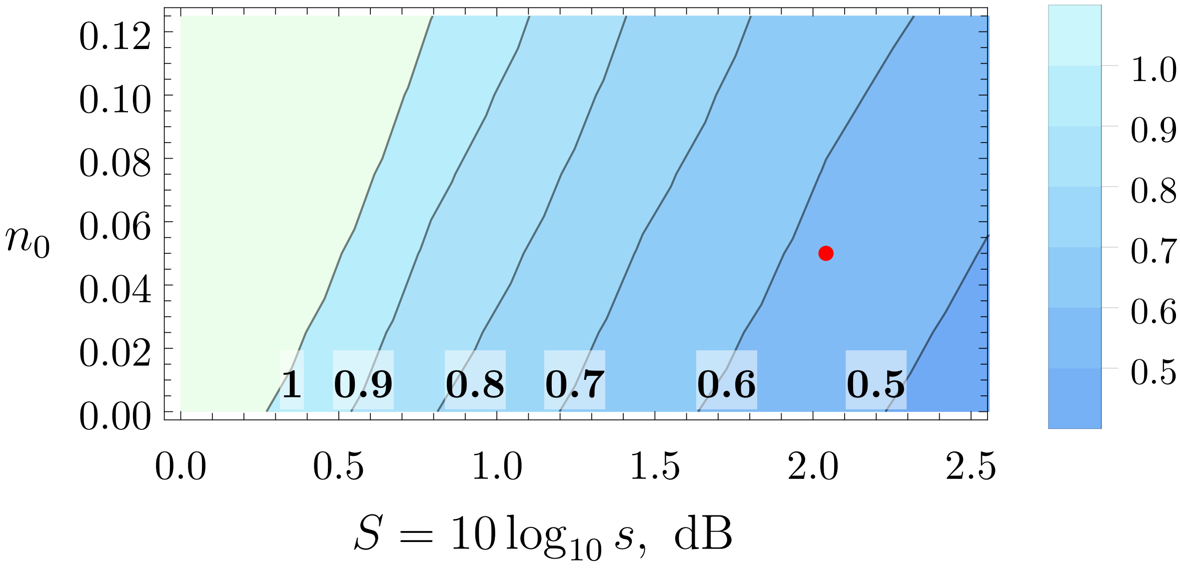

FIG. S1. The dependence of the attainable nonlinear variance $\min_\lambda\ [\mathrm{Var}(p-\lambda x^2)]$ (in units of shot noise) on the squeezing s and occupancy n_0 of the initial state of the particle (7). Total strength of the nonlinearity equals $\gamma = 0.05$. The red dot at $(2, 0.05)$ corresponds to the conditions of Fig. 2 of the main text.

the variance much below 1. For the grid functions this Gaussian turns into a Dirac δ, which maps the initial WF onto itself. Importantly, for the values of variance which are smaller than the step of the grid, the errors introduced in computation of the convolution will overwhelm the impact of the thermal bath.

For Figs. 2 and S1 we use $\sigma_{\mathrm{th}}\theta^2 = 0.03$ which, for an oscillator of eigenfrequency $\Omega_m = 2\pi \times 100\,\mathrm{kHz}$ and $Q = 10^6$ is equivalent to occupation of the environment equal to $n_{\mathrm{th}} \approx 10^8$ phonons. This is the equilibrium occupation of such an oscillator at the temperature of 500 K.

Squeezing of the initial state.—Presqueezing of the initial vacuum state of the mechanical oscillator can help to enhance the attainable cubic (and higher order) nonlinearity. It is apparent in the case of application of a cubic gate, when de-amplification and amplification manipulate directly with γ. That is, a CV cubic gate transforms a state described by quadratures x and p into one with quadratures x and $p+\gamma x^2$. If this state is squeezed in p direction before the gate and after the gate it is antisqueezed in the same direction, the quadrature evolve as follows

$$\begin{pmatrix} x \\ p \end{pmatrix} \to \begin{pmatrix} sx \\ s^{-1}p \end{pmatrix} \to \begin{pmatrix} sx \\ s^{-1}p + s^2\gamma x^2 \end{pmatrix} \to \begin{pmatrix} x \\ p + s^3\gamma x^2 \end{pmatrix}. \tag{S11}$$

We prove that squeezing the initial state helps to enhance the nonlinear squeezing by Fig. S1 where we plot contours of the achievable nonlinear squeezing over the space of the initial occupation numbers n_0 and squeezing s of the initial squeezed thermal state (7). The effect of squeezing is equivalent to cooling in this sense.